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SUMMABILITY IN BANACH LATTICES

BY

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The M- and L-spaces are characterized by their properties of summability. An example of Banach space which cannot be realised as a Banach lattice is presented also.

1. INTRODUCTION

In [5] Dvoretzky and Rogers remarked that a Banach space Z which verifies the condition $l^1[Z] = l^1\{Z\}$ (we use the Pietsch's notations in [25]) is isomorphic to a space of finite dimension. A more precise result was obtained by Grothendieck [8]: a locally convex Fréchet vector space X is nuclear if and only if $l^1[X] = l^1\{X\}$. In the present paper we split Grothendieck's condition of summability by giving suitable conditions for the spaces $L^p(\mu)$ (μ a positive Radon measure and $1 \leqslant p < \infty$) and $C(S)$. Our results are motivated by the local structure of nuclear lattices. See section 5 below for details.

Before stating explicitely these results we shall define the terms appearing.

By a Banach lattice we shall mean a Banach space which is also a vector lattice and in addition:

$$|x| \leqslant |y|, \quad \text{implies} \quad \|x\| \leqslant \|y\|.$$

An example of Banach space which cannot be realised as Banach lattice will be presented in the Appendix.

We shall denote by $l^p\{X\}$ (X a Banach space and $1 \leqslant p < \infty$) the vector space of all sequences $\{x_n\}$ of elements of X such that:

$$\Sigma \|x_n\|^p < \infty$$

For all the problems concerning the summability or nuclearity we refer to [8] and [25].

Other notations:

$\mathcal{L}(E, F)$ = the vector space of all continuous linear operators from E to F. Here E and F are two locally convex Hausdorff spaces

$E^* = \mathcal{L}(E, \mathbb{R})$

$C(S)$ = the Banach lattice (with the sup norm) of all continuous real functions defined on the compact Hausdorff space S.

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$L^p(\mu)$ = the completion $\left(\text{with respect to the norm } \|f\|_p = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}}\right)$ of the space of all continuous real functions $f: S \to \mathbb{R}$ having a compact support. Here μ denotes a positive Radon measure defined on the locally compact Hausdorff space S

$$l^p = l^p\{\mathbb{R}\}, \qquad 1 \leqslant p < \infty,$$

We can now present a brief survey of the main results obtained.

THEOREM. *A Banach lattice Z is algebraically topologically lattice isomorphic to a space $L^p(\mu)$ if and only if it satisfies the following statements* :

$(L^p.\ a)$ $\qquad \{x_n\}_n \in l^1[Z], \quad x_n \geqslant 0$ implies $\{x_n\}_n \in l^p\{Z\} \cap l^1(Z)$

$(L^p.\ b)$ $\qquad \| \Sigma\, x_i\|^p \leqslant \gamma \Sigma \, \|x_i\|^p$

for every finite family $\{x_i\}_i$ of disjoint elements of Z, γ being a positive constant which depends only on Z.

As a consequence of this result we shall prove (Remark 3.7 below) the lattice invariance principle for Hilbert spaces.

The question of summability of certain kinds of sequences in a space $L^p(\mu)$ has been considered before by various authors. Here we mention only Orlicz's result in [24]: If $\{x_n\}_n$ is a summable sequence in $L^p(\mu)$ then $\sum_{n=1}^{\infty} \|x_n\|^{\gamma(p)} < \infty$ where $\gamma(p) = 2$ if $1 \leqslant p \leqslant 2$ and $\gamma(p) = p$ if $p \geqslant 2$. The exponents $\gamma(p)$ are the best possible i.e. they cannot be replaced by smaller constants.

Another class of Banach lattices which admit a characterization by summability is the class of all order σ-complete Banach lattices having an order continuous topology. In the next section we shall prove the following result :

THEOREM. *For an E ordered Banach space which is also a σ-complete vector lattice the following statements are equivalent* :

(i) $x_n \downarrow 0$ *(in order) implies $x_n \to 0$ topologically.*

(ii) *Every order interval in E is relatively weakly compact.*

(iii) $\{x_n\}_n \in l^1_{\sigma}[E] \quad x_n \geqslant 0$ *implies* $\{x_n\}_n \in l^1(E)$.

Here $l^1_{\sigma}[E]$ denotes the vector space of all $\{x_n\}_n \in l^1[E]$ such that for a suitable $x \in E$, $x > 0$, we have :

$$\sum_{n \in F} |x_n| \leqslant x,$$

whenever $F \subset \mathbb{N}$ a finite subset.

Concerning the M-spaces i.e. the closed vector sublattices of a space $C(S)$ it was conjectured [1] that they are characterized (up to an algebraic topologic lattice isomorphism) by the following condition

(aM) $\qquad \{x_n\}_n \in l^1[E]$ implies $\{|x_n|\}_n \in l^1[E]$.

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[1] See [11]

One can show (see section 4 below) that this problem is equivalent to the following : Let E be a Banach lattice. Suppose that every positive operator $T \in \mathcal{L}(E, l^1)$ is absolutely summing. Need every positive operator $T \in \mathcal{L}(E, l^1)$ be nuclear?

An aM-space i.e. a Banach lattice which verifies the condition (aM) above, cannot be isomorphic to a space l^p $(1 \leqslant p < \infty)$. The conjugate of an aM-space is a Banach lattice with an order continuous topology. See section 4 below for details.

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2. BANACH LATTICES HAVING AN ORDER CONTINUOUS TOPOLOGY

A Banach lattice E is said to have an *order continuous topology* if :

$$x_n \downarrow 0 \text{ (in order) implies } \quad x_n \to 0 \quad \text{topologically.}$$

The usual examples are the space $L^p(\mu)$ $(1 \leqslant p < \infty)$ and c_0, the Banach lattice of all null sequences of real numbers. A characterization for all the σ-complete Banach lattices having an order continuous toplogy was obtained in [21]. In the next we obtain this characterization by using the order continuous operators :

2.1. PROPOSITION. *Let X, Z be two ordered Banach spaces, Z being supposed in addition a σ-complete vector lattice. Then for $U \in \mathcal{L}(Z, X)$ $U \geqslant 0$, the following statements are equivalent :*

(a) U maps the order intervals of Z into relatively weakly compact subsets of X.

(b) If $\{z_n\}_n$ is a decreasing sequence of positive elements of Z then $\{Uz_n\}_n$ is a convergent sequence.

Proof. For $z \in Z$, $z \geqslant 0$, consider the following vector space :

$$Z_z = \{y \in Z \,;\ |y| \leqslant \lambda z \text{ for suitable } \lambda > 0\},$$

normed by :

$$\|y\|_z = \inf \{\lambda > 0 \,;\ |y| \leqslant \lambda z\}.$$

Then Z_z is an abstract M-space in the terminology of Kakutani [13], and therefore it is isometric and lattice isomorphic to a space $C(S_z)$ for S_z a suitable compact Hausdorff space. Denote by U_z the following product of operators :

$$Z_z \xrightarrow{i_z} Z \xrightarrow{U} X.$$

Here i_z denotes the canonical embedding.

The assertion (a) above is equivalent to the following :

(a') The operators U_z, $z \geqslant 0$, are all weakly compact.

Then the equivalency (a′) $\Leftrightarrow$ (b) follows immediately from an earlier result due to Grothendieck [7] Theorem 6.

For a $\Rightarrow$ b we present a direct proof. Let $\{z_n\}_n$ be a decreasing sequence of positive elements of Z. Since $U \geqslant 0$ it follows that $\{Uz_n\}_n$ is a weakly Cauchy sequence in Z. On the other hand (a) implies that $\{Uz_n\}_n$ is contained in a weakly compact subset of Z. Therefore $\{Uz_n\}_n$ is a weakly convergent sequence of positive elements of Z. It is also a decreasing sequence and thus the generalized Dini's theorem (see [29] ch. V, 4.3) implies that this sequence is also convergent, q.e.d.

2.2. COROLLARY. *Let Z be an ordered Banach space which is also a σ-complete vector lattice. The following statements are equivalent :*

(1) *Every order interval of Z is relatively weakly compact.*

(2) *The topology of Z is order continuous.*

2.3. *Remark.* The conditions (1) and (2) above are equivalent to the following :

$$(3) \qquad \{x_n\}_n \in l^1_{\alpha}[Z], \quad x_n \geqslant 0 \text{ implies } \{x_n\}_n \in l^1(Z)$$

In fact Orlicz's result in [23] concerning the unconditionally weakly convergent series [2] shows that (1) $\Rightarrow$ (3). On the other hand (3) $\Rightarrow$ (2), since if we consider a sequence $x_n \in Z$, $x_n \geqslant 0$ and $x_n \downarrow 0$ (in order), then the series $(x_1 - x_2) + (x_2 - x_3) + \ldots$ defines an element of $l^1_{\alpha}[Z]$, q.e.d.

2.4. *Remark.* Let Z be an order σ-complete Banach lattice which is also a separable dual. Then Z has order continuous topology. In fact, it was remarked in [7] that every $T \in \mathrm{L}\,(C(S), Z)$ is a weakly compact operator whenever S is a compact Hausdorff space. By considering the spaces Z_z as above it follows that every order interval in Z is relatively weakly compact q.e.d.

3. CHARACTERIZATION OF $L^p(\mu)$ BY SUMMABILITY

A Banach lattice E is said to be *an abstract L^p-space* $(1 \leqslant p < \infty)$ if the topology and the order are related by :

$$x, y \in E \quad \inf(|x|, |y|) = 0 \quad \text{implies} \quad \|x + y\|^p = \|x\|^p + \|y\|^p.$$

Marti [18] and Bernau [34] have remarked that every abstract L^p-space is equivalent to a space $L^p(\mu)$ for μ a suitable positive Radon measure. However several special cases were proved earlier. For example the case $p = 1$ was illuminated by Kakutani [12]. Bohnenblust [2] considers the case of the separable Banach lattices order σ-complete.

Joint characterizations of L^p- and M-spaces were obtained in [2], [32], etc.

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[2] We recall this result : Let X be a Banach space and $\{x_n\}_n$ a sequence of elements of X. Suppose that for every increasing sequence n_i, $i \in \mathbb{N}$, of integers the series Σx_{n_i} is weakly convergent. Then for every increasing sequence n_i, $i \in \mathbb{N}$, of integers the series Σx_{n_i} is convergent in X.

In the next the abstract L^p-spaces are characterized by their summability properties. The case $p = 1$ was treated independently by Schlotterbeck [30]:

3.1. THEOREM. *A Banach lattice E is topologically algebraically lattice isomorphic to an abstract L^p-space if, and only if, E satisfies the following conditions:*

$(L^p.a)$ $\quad \{x_n\}_n \in l^1[E], \quad x_n \geqslant 0 \quad$ implies $\quad \{x_n\}_n \in l^p\{E\} \cap l^1(E)$

$(L^p.b)$ $\quad \| \Sigma x_i \|^p \leqslant \gamma^p \Sigma \|x_i\|^p,$

for every finite family $\{x_i\}_i$ of disjoint elements of E, γ being a positive constant which depends only of E.

Proof. The necessity. From Orlicz's result cited above it follows that:

$$l^1[L^p(\mu)] = l^1(L^p(\mu)).$$

On the other hand for every finite family of positive elements of $L^p(\mu)$ we have $\sum_{i=1}^{n} \int f_i^p \,d\mu \leqslant \int \left(\sum_{i=1}^{n} f_i \right)^p d\mu,$
which implies $(L^p.a)$.

The necessity of $(L^p.b)$ is obvious.

The sufficiency will be proved in three steps.

A. First observe that E is order σ-complete. In fact let us consider an increasing sequence $x_n \in E$, $x_1 = 0$ and $x_n \leqslant x$ for some $x \in E$. Then $\{x_{n+1} - x_n\}_n$ is a weakly summable sequence and therefore (see $(L^p.a)$) it belongs to $l^1(E)$, which implies that $\{x_n\}_n$ is a convergent sequence. Therefore E is order σ-complete and its topology is order continuous.

B. Let us suppose that E has a Freudenthal unit $u > 0$ i.e.

$$x \in E, \ \inf\,(|x|, u) = 0 \ \text{implies} \ x = 0.$$

Denote by $\mathscr{B}(E)$ the set of all $e \in E$ such that $\inf(e, u - e) = 0$ and by $\mathscr{S}(E)$ the set of all elements of the form:

$$x = \sum_{i \in F} \alpha_i \, e_i,$$

where $\alpha_i \in \mathbb{R}$, $e_i \in \mathcal{B}(E)$, $\inf\,(e_i, e_j) = 0$ for $i \neq j$, F an arbitrary finite set. In the next we shall consider only such representations for the elements of $\mathscr{S}(E)$.

Denote by $\| \quad \|$ the original norm on E. In the next we shall show that E can be renormed by

$$|||x||| = \gamma \sup \Big(\sum_{i \in F} \alpha_i^p \|e_i\|^p\Big)^{1/p},$$

where the sup is taken over all the elements $z = \sum_{i \in F} \alpha_i e_i \in \mathscr{S}(E)$, such that $0 \leqslant z \leqslant |x|$. We have:

(1) $x \in E, \alpha \neq 0$ implies $|||\alpha x||| = |\alpha| \cdot |||x|||$,

(2) $x, y \in E$, $\inf(|x|, |y|) = 0$ implies $|||x|||^p + |||y|||^p \leqslant |||x+y|||^p$.

Moreover there exists $c > 0$ such that:

(3) $$|||\quad||| \leqslant c\|\quad\|.$$

In fact, if the contrary is true then there exists a sequence $x_n \in E$ such that $\|x_n\| = 1$ and $|||x_n||| \geqslant 2^n\gamma$. This implies the existence of a sequence $s_n \in \mathscr{S}(E)$ with the following properties:

$$s_n = \sum_{i=1}^{p(n)} \alpha_{in} e_{in}$$

$$0 \leqslant s_n \leqslant |x_n|$$

$$\sum_{i=1}^{(p)n} \alpha_{in}^p \|e_{in}\|^p \geqslant 2^{np}.$$

Then for $f \in E^*$, $f^* \geqslant 0$ we have:

$$\sum_{i=1}^{\infty} \sum_{i=1}^{p(n)} \frac{\alpha_{in}}{2^n} f(e_{in}) \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} f(|x_n|) \leqslant \|f\|,$$

i.e. the sequence $\left\{\frac{\alpha_{in}}{2^n} e_{in}\right\}_{i,n}$ is weakly summable. On the other hand:

$$\sum_{n=1}^{\infty} \sum_{i=1}^{p(n)} \left(\frac{\alpha_{in}}{2^n}\right)^p \|e_{in}\|^p = \infty,$$

which contradicts $(L^p.a)$ and thus (3) follows. Particularly $|||\quad|||$ is finite. Then from (1) it follows that:

(4) $$|||\alpha x||| = |\alpha| \cdot |||x|||,$$

for every $\alpha \in \mathbb{R}$, $x \in E$. We have also:

(5) $$\|\quad\| \leqslant |||\quad|||.$$

In fact, from $(L^p.b)$ it follows that this inequality holds for the elements of $\mathscr{S}(E)$. Let $x \in E$ and $\varepsilon > 0$. It was remarked by Freudenthal in [6] that there exists an $s \in E$ such that:

$$0 \leqslant x - s \leqslant \varepsilon u,$$

where $s = \sum_{n=1}^{\infty} \alpha_n e_n$, $\alpha_n \geqslant 0$, $e_n \in \mathcal{B}(E)$, $\inf(e_m, e_n) = 0$ for $m \neq n$, the series being order convergent. Since the topology of E is order continuous, this series is also $\| \ \|$-convergent. Then from $(L^p.b)$ it follows that :

$$\|x\| \leqslant \left\|x - \sum_{n=1}^{\infty} \alpha_n e_n\right\| + \left\|\sum_{n=N+1}^{\infty} \alpha_n e_n\right\| + \left\|\sum_{n=1}^{N} \alpha_n e_n\right\| \leqslant$$

$$\leqslant \|x - s\| + \left\|\sum_{n=N+1}^{\infty} \alpha_n e_n\right\| + |||x||| \leqslant \varepsilon \|u\| + |||x||| + \left\|\sum_{n=N+1}^{\infty} \alpha_n e_n\right\|,$$

and (5) follows.

Finally we shall prove that :

(6) $\qquad x, y \in E \quad$ implies $\quad |||x + y|||^p \leqslant |||x|||^p + |||y|||^p.$

Then from (2) — (6) it follows immediately that E is isomorphic to an abstract L^p-space.

In order to prove (6) let us consider an $e \in \mathcal{B}(E)$, $e \neq 0$ such that :

$$e \leqslant |x| + |y|.$$

Since E is a σ-complete lattice with unit for each $z > 0$ there exists $e \in \mathcal{B}(E)$, $e \neq 0$ and $\alpha > 0$ such that $z \geqslant \alpha e$. Then for $0 < \varepsilon < 1$ fixed there exist $e' \in \mathcal{B}(E)$, $0 < e' \leqslant e$ and two positive numbers $\alpha_{e'}$ and $\beta_{e'}$ such that :

$$\alpha_{e'} \ e' \leqslant |x|,$$

$$\beta_{e'} \ e' \leqslant |y|,$$

and $\alpha_{e'} + \beta_{e'} \geqslant 1 - \varepsilon$. Let us denote by $\mathcal{E}$ the set of all such e'. By using Zorn's lemma and $(L^p.a)$ we can consider a maximal subset $\mathcal{F}$ of $\mathcal{E}$ such that :

$$e', e'' \in \mathcal{F}, \ e' \neq e'' \quad \text{implies} \quad \inf(e', e'') = 0.$$

Then for $f \in E^*$, $f^* > 0$ we have :

$$f(\sum_{e' \in F} \alpha_{e'} e') \leqslant f(|x|)$$

$$f(\sum_{e' \in F} \beta_{e'} e') \leqslant f(|y|),$$

for every $F \subset \mathcal{F}$ a finite subset. From $(L^p.a)$ it follows that $\alpha_{e'}, \beta_{e'} = 0$ except at most countable subset of $\mathcal{F}$. Thus we can consider the following element of E:

$$z = \sum_{e' \in \mathcal{F}} (\alpha_{e'} + \beta_{e'})e' = \sup \{(\alpha_{e'} + \beta_{e'})e' \ ; \ e' \in \mathcal{F}\}. \tag{7}$$

The maximality of $\mathcal{F}$ implies that:

$$z \geqslant (1 - \varepsilon)e. \tag{8}$$

In fact, let us suppose that the contrary is true. Since $\alpha_{e'} + \beta_{e'} \geqslant 1 - \varepsilon$ for each $e' \in \mathcal{F}$ it follows that:

$$f = \sup \{e' ; \quad e' \in \mathcal{F}\} \not\geqslant e.$$

Or $f \in \mathcal{B}(E)$ and in addition

$$e - \inf(e, f) \in \mathcal{B}(E)$$

$$e - \inf(e, f) \neq 0 \qquad ,$$

$$\inf \{[e \dot{-} \inf(e, f)], f\} = 0$$

which contradicts the maximality of $\mathcal{F}$. Then (6) follows from (8) and the following two inequalities:

$$\Sigma \alpha_{e'} e' \leqslant |x|$$

$$\Sigma \beta_{e'} e' \leqslant |y|.$$

C. Let us consider the general case. Denote by $\{e_i\}_{i \in I}$ a maximal system of elements of E which satisfy $\inf (e_i, e_j) = 0$ for $i \neq j$. It is known that

$$|x| = \sup \{[e_i] (|x|) \ ; \ i \in I\},$$

for each $x \in E$. Here $[e_i]$ denotes the proiector generated by e_i.

Then:

$$[e_i] (|x|) = \sup \{\inf (|x|, ne_i) ; \ n \in \mathbb{N}\},$$

for each $x \in E$.

For each $i \in I$ consider the following subspace of E:

$$E_i = \{x \in E ; \ \inf (|x|, |z|) = 0 \text{ if } \inf (|z|, e_i) = 0\}$$

is a σ-complete Banach lattice with unit e_i. Let $|||\ |||_i$ be the norm defined as above on E_i. We shall show that E can be renormed by:

(9) $$|||x||| = \Big(\sum_{i\in I} |||[e_i](|x|)|||_i^p\Big)^{1/p}.$$

First observe that:

(10) $$|||\ |||_i \leqslant c\|\ \|,$$

for some $c > 0$. In fact, if the contrary is true, then for every $n \in \mathbb{N}$ there exist $i(n) \in \mathbb{N}$ and $x_n \in E_{i(n)}$ such that $\|x_n\| = 1$ and $|||x_n|||_{i(n)} > 2^n\gamma$. Then there exists also a sequence $s_n \in \mathcal{S}(E_{i(n)})$ such that:

$$0 \leqslant s_n = \sum_{k=1}^{r(n)} \alpha_{kn} e_{kn} \leqslant |x_n|$$

$$e_{kn} \in \mathcal{B}(E_{i(n)}),\ \inf(e_{in}, e_{jn}) = 0 \text{ for } i \neq j$$

$$\sum_{k=1}^{r(n)} \alpha_{kn}^p \|e_{kn}\|^p \geqslant 2^{np}.$$

Then the sequence $\left\{\frac{1}{2^n}\alpha_{kn}e_{kn}\right\}_{k,n}$ is weakly summable. In fact for each $f \in E^*$, $f \geqslant 0$ we have:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{r(n)} \frac{1}{2^n}\alpha_{kn} f(e_{kn}) \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n}(|x_n|) \leqslant \|f\|.$$

On the other hand:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{r(n)} \frac{1}{2^{pn}}\alpha_{kn}^p \|e_{kn}\|^p = \infty,$$

which contradicts $(L^p\cdot a)$ and (10) follows.

The series appearing in (9) is convergent (i.e. $|||\ |||$ is really a norm). For this, remark that from $(L^p\cdot a)$ it follows the convergence of the series

$$\sum_{i\in H} \|[e_i](|x|)\|^p,$$

whenever H at most countable subset of I.

Denote by $\mathcal{S}(E)$ the linear hull of the set:

$$\mathcal{B}(E) = \bigcup_{i\in I} \mathcal{B}(E_i).$$

Then $\mathcal{S}(E)$ is $\|\ \|$-dense in E.

The norms $\| \ \|$ and $||| \ |||$ are equivalent on $\mathscr{S}(E)$. It is clear that $\| \ \| \leqslant ||| \ |||$. Moreover there exists $k > 0$ such that :

$$||| \ ||| \leqslant k \| \ \|.$$

In fact, if the contrary is true, there exists a sequence $x_n \in \mathscr{S}(E)$ such that :

$$\|x_n\| = 1,$$

$$|||x_n||| > 2^n,$$

$$|x_n| = \sum_{i \in F_n} [e_i](|x_n|),$$

where F_n is a finite subset. We have also :

$$\sum_{i \in F_n} |||[e_i](|x_n|)|||^p_i \geqslant 2^{np}.$$

Then the sequence $\left\{\frac{1}{2^n} [e_i](|x_n|)\right\}_{i,n}$ is weakly summable but not p-absolutely summable, which contradicts $(L^p \cdot a)$, q.e.d.

3.2. Corolarry. *For E a Banach lattice the following statements are equivalent* :

(a) *E is isomorphic to an AL-space.*

(b) $\{x_n\}_n \in l^1[E]$, $x_n \geqslant 0$ implies $\{x_n\}_n \in l^1\{E\}$.

(c) $\{x_n^{\cdot}\}_n \in l^1(E)$, $x_n \geqslant 0$ implies $\{x_n\}_n \in l^1\{E\}$.

(d) $\{x_n\}_n \in l^1_{\sigma}[E]$, $x_n \geqslant 0$ implies $\{x_n\}_n \in l^1\{E\}$.

(e) *Every E-valued σ-additive positive measure defined on a σ-algebra is of finite variation.*

(f) *For every M-space Z we have* :

$T \in \mathscr{L}(Z, E)$, $T > 0$ *implies* $T =$ *absolutely summing*

(g) *As (f) with $T =$ majorized.*

(h) *As (f) with $T =$ integral.*

The proof of (a) $\Leftrightarrow$ (d) is similar with the proof of Theorem 3.1 above and we omit it. Every M-space is also an aM-space and thus (a) $\Rightarrow$ (f). From [25] 2.3.4 it follows easily that (f) $\Rightarrow$ (g). Clearly (h) $\Rightarrow$ (g). It was remarked by Singer [31] that every majorized operator defined on a space $C(S)$ is also integral. The bidual of an M-space is a space $C(S)$ and every majorized operator is weakly compact. Then (g) $\Rightarrow$ (h).

For every Banach Lattice E there exists an order isomorphism :

$$\chi : l^1[E] \to \mathscr{L}(c_0, E)$$

defined by:

$$\chi(\{x_n\})(\{\alpha_n\}) = \sum_{n=1}^{\infty} \alpha_n x_n.$$

An operator $T \in \mathcal{L}(c_0, E)$ is absolutely summing if and only if $\chi^{-1}(T) \in \in l^1\{E\}$. Then (f) $\Rightarrow$ (b).

Clearly (f) $\Rightarrow$ (e).

(e) $\Rightarrow$ (c). Remark that every $\{x_n\}_n \in l^1(E)$, $x_n \geqslant 0$ generates a σ-additive positive measure $m: \mathcal{P}(\mathbb{N}) \to E$ defined by:

$$m(A) = \sum_{n \in A} x_n,$$

whenever $A \subset \mathbb{N}$.

(c) $\Rightarrow$ (f) follows from [25] 2.1.2 and the following remark. For every M-space Z we have:

$$\{x_n\}_n \in l^1(Z) \quad \text{implies} \quad \{|x_n|\} \in l^1(Z),$$

q.e.d.

3.3. *Remark.* For $1 < p < \infty$ the condition $(L^p \cdot a)$ is only necessary. In fact every p-absolutely summing operator is also q-absolutely summing for every $q \geqslant p$.

The fact that $(L^p \cdot a)$ suffices for $p = 1$ implies that there exists no Banach lattices which verify $(L^p \cdot a)$ for some $0 < p < 1$.

3.4. *Remark.* The condition $(L^p \cdot b)$ holds in every M-space.

3.5. *Remark.* Generally $l^1[E] \cap l^p\{E\}$ is not contained in $l^1(E)$ if $1 < p < \infty$. A simple counter-example can be obtained for $E = c_0$ as follows. Let $\{\alpha_n\}_n \in l^2$, $\alpha_1 = 2$, $\alpha_n \geqslant 0$ and $\Sigma \alpha_n = \infty$. Consider a disjoint decomposition $\{F_n\}_n$ of $\mathbb{N}$ such that:

$$\begin{aligned} F_1 &= \{1, 2, \ldots\ldots, n_1\} \\ F_2 &= \{n_1 + 1, \ldots, n_2\} \\ &\ldots\ldots \end{aligned},$$

and:

$$1 \leqslant \sum_{i \in F_n} \alpha_i < \sum_{i=1}^{\infty} \alpha_i^2$$

for every $n \in \mathbb{N}$. Denote by e_n the n-th coordinate wise sequence in c_0. Then the following sequence of elements of c_0:

$$x_i = \alpha_i \, e_n$$

if $i \in F_n$, belongs to $l^1[c_0] \cap l^2\{c_0\}$. Clearly $\{x_n\}_n \notin l^1(c_0)$, q.e.d.

3.6. *Remark.* Let E be a Banach lattice order complemented in some L^1-space Z i.e. there exist two continuous positive operators $P \in \mathcal{L}(Z, E)$

and $Q \in \mathcal{L}(E, Z)$ such that $PQx = x$ for each $x \in E$. Then E is algebraically topologically lattice isomorphic to an L^1-space. For the proof see 3.2 (f) above.

A stronger version of this result is the following minimality principle for L^1-space:

Let E be a Banach lattice such that there exists an algebraic topological isomorphism T from E into suitable $L^1(\mu)$. Suppose that $T > 0$. Then from 3.2(f) above it follows that E is order isomorphic to suitable L^1-space.

A more precise result holds for Hilbert spaces:

3.7. *Remark* (*Lattice invariance of Hilbert spaces*). *Let E be a Hilbert space which is also a Banach lattice*[3]. *Then E is algebraically topologically lattice isomorphic to a suitable L^2-space.*

Proof. By using a standard argument we may assume that E has a Freudenthal unit i.e. E is a separable Hilbert space. Then our corollary follows from Theorem 3.1 above:

a) E verifies ($L^2.a$). In fact, E being reflexive we have $l^1(E) = l^1[E]$. Orlicz's result cited in Introduction shows that $l^1(E) \subset l^2\{E\}$.

b) E verifies ($L^2.b$). In a Hilbert space all the unconditional bases are equivalent (e.g. see [18]) and therefore for every such a basis $\{e_n\}$ there exists a positive constant γ such that:

$$(*) \qquad \left\| \sum_{n=1}^{\infty} \alpha_n e_n \right\|^2 \leqslant \gamma \sum_{n=1}^{\infty} |\alpha_n|^2 \, \|e_n\|^2,$$

for every $\{\alpha_n\}_n \in l^2$. We can choose a common $\gamma > 0$ and this fact clearly implies ($L^2.b$). Indeed, if the contrary is true for every $n \in \mathbb{N}$ we can find a sequence $\{\alpha_{i,n}\}_i \in l^2$ and an unconditional basis $\{e_{i,n}\}_i$ such that:

$$\left\| \sum_{i=1}^{\infty} \alpha_{i,n} e_{i,n} \right\|^2 > 2^{2n} \sum_{i=1}^{\infty} |\alpha_{i,n}|^2 \, \|e_{i,n}\|^2.$$

Then $\{e_{i,n}\}_{i,n}$ is an unconditional basis in the Hilbert space:

$$\bigoplus_{i=1}^{\infty} E = \left\{ f : \mathbb{N} \to E; \ \sum_{n=1}^{\infty} \|f(n)\|^2 < \infty \right\},$$

for which no relation of the form (*) holds, q.e.d.

4. *aM* AND *M*-SPACES

Let us recall that by an ***aM***-space we mean a Banach lattice E which verifies the following condition:

$$(aM) \qquad \{x_n\}_n \in l^1[E] \text{ implies } \{|x_n|\}_n \in l^1[E].$$

[3] This means that E is algebraically topologically isomorphic to a Banach lattice.

Every M-space is also an aM-space. In fact by using a result due to Kakutani [13] it suffices to consider only the case $E = C(S)$. Then our assertion follows immediately by using the Dirac measures.

In connection with the condition ($L^p.a$) above we wish to point out that a similar condition holds for the M-spaces. We need a definition. A sequence $\{x_n\}_n$ of elements of E is called p-weakly summable if

$$\sum_{n=1}^{\infty} \langle x_n, x^* \rangle|^p < \infty,$$

for every $x^* \in E^*$. Denote by $l^p[E]$ the vector space of all such sequences.

4.1. Proposition. *Let E be an aM-space. Then for $1 < p < \infty$ we have*

$$\{x_n\}_n \in l^p[E] \text{ implies } \{|x_n|\}_n \in l^p[E].$$

Proof. Let $\{x_n\}_n \in l^p[E]$ and $q \in \mathbf{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for every $\{\alpha_n\}_n \in l^q$ we have $\{\alpha_n x_n\}_n \in l^1[E]$. From (aM) it follows that $\{|\alpha_n| \cdot |x_n|\}_n \in l^1[E]$ and therefore $\{x_n\}_n \in l^p[E]$ q.e.d.

An interesting but unsolved problem is the following:

4.2. Problem. *Need every aM-space be isomorphic to an M-space?*

This problem can be translated in language of operators as it follows: Let Z be a Banach lattice such that every $L^1(\mu)$-valued positive operator defined on Z is absolutely summing. Need every positive operator of $\mathcal{L}(Z, L^1(\mu))$ be integral? This is an easy consequence of the following two propositions:

4.3. Proposition. *For E a Banach lattice the following statements are equivalent:*

(a) *E is order isomorphic to an M-space*

(b) *For every positive Radon measure μ we have:*

$$T \in \mathcal{L}(E, L^1(\mu)),\ T \geqslant 0 \textit{ implies } T = \textit{integral}$$

(c) $T \in \mathcal{L}(E, l^1)$, $T \geqslant 0$ *implies* $T =$ *nuclear.*

Proof. (a) $\Rightarrow$ (b). Let $T \in \mathcal{L}(E, L^1(\mu))$, $T \geqslant 0$. The conjugate of E is order isomorphic to a space $L^1(\nu)$ and thus $T^* \in \mathcal{L}(L^\infty(\mu), L^1(\nu))$ is absolutely summing. On the other hand it was remarked by Singer [31] that every majorized operator defined on a space $C(S)$ is also integral. Particularly T^* is integral, etc.

(b) $\Rightarrow$ (c). Since l^1 is a separable dual every integral operator $T \in \mathcal{L}(E, l^1)$ is also nuclear. See [8] §4, n° 3, Corollaire 3.

(c) $\Rightarrow$ (a). For every $\{x_n^*\}_n \in l^1[E^*]$, $x_n^* > 0$, we can consider the following positive operator $T \in \mathcal{L}(E, l^1)$ given by:

$$T(x) = \{\langle x, x_n^* \rangle\}_n.$$

From hypothesis, T is nuclear and therefore integral. An $L^1(\mu)$-valued integral operator maps normed bounded subsets into order bounded subsets. See [8] Theorem 11 for details. Therefore $\{x_n^*\}_n \in l^1\{E^*\}$ and thus (see Theorem 3.1) E^* is order isomorphic to an L^1-space, q.e.d.

The next result gives a simple characterization for aM-spaces by using the absolutely summing operators:

4.4. PROPOSITION. *A Banach lattice E is order isomorphic to an aM-space if, and only if, every $L^1(\mu)$-valued positive operator defined on E is absolutely summing.*

Proof. The necessity is clear. The sufficiency. First remark that every $x^* \in E^*$, $x^* > 0$ defines naturally a relation of equivalency:

$$x \sim y \text{ if, and only if, } x^*(|x-y|) = 0.$$

The completion of $E/\sim$ with respect to the norm:

$$\|x\|_{x^*} = x^*(|x|)$$

is an L-space denoted by $L^1(x^*)$. From hypothesis it follows that the canonical mappings $E \to L^1(x^*)$ are all absolutely summing which implies that E is an aM-space, q.e.d.

4.5. *Remark.* Let E be a Banach lattice order complemented in some M (respectively aM)-space. Then E is order isomorphic to an M-(aM-) space. More generally let E be a Banach lattice and Z an M- (or an aM-) space such that there exist two continuous operators $P \in \mathcal{L}(Z, E)$, $Q \in \mathcal{L}(E, Z)$ such that $P \geqslant 0$ and $PQx = x$ for every $x \in E$. Then E is order isomorphic to an M-(aM-) space.

Finally we state three useful remarks about the aM-spaces.

4.6. *Remark.* The dual of an aM-space E is a Banach lattice having order continuous topology. This follows from 2.3 above. In fact every $\{x_n^*\}_n \in l^1[E^*]$, $x_n^* > 0$, defines an absolutely summing operator $T \in \mathcal{L}(E, l^1)$ by the formula:

$$T(x) = \{\langle x, x_n^* \rangle\}_n.$$

Every absolutely summing operator is weakly compact ([8] § 4, n° 6 Lemme 17) and thus the restriction of T^* to c_0 is also weakly compact. Clearly this restriction is given by:

$$T^*(\{a_n\}) = \sum_{n=1}^{\infty} a_n x_n^*.$$

From Orlicz's result cited in section 2 above it follows that $\{x_n^*\} \in l^1(E^*)$ q.e.d.

4.7. *Remark.* An aM-space E cannot be order isomorphic to a space l^p. Here we consider only the case $2 \leqslant p < \infty$. For $1 \leqslant p \leqslant 2$ see [11]. Suppose that E satisfies to (aM). Since the canonical mapping $i_p: l^1 \to l^p$ is absolutely summing ([25] 2.4.2) it follows that:

$$T \in \mathcal{L}(l^p, l^1),\ T \geqslant 0 \text{ implies } T \circ i_p = \text{nuclear}.$$

In fact, the product of two absolutely summing operators is a nuclear. See [8] Theorem 14 for the proof.

On the other hand (see [25] 3.1.10) an operator $T \in \mathcal{L}(l^1, l^1)$ given by a matrix $\{a_{ij}\}_{i,j}$ is nuclear if, and only if :

$$\sum_{j=1}^{\infty} \sup_{i \in \mathbb{N}} |a_{ij}| < \infty,$$

Consider a sequence $\{a_i\}_i \in l^q$, $a_i > 0$, and $\Sigma a_i = \infty$. It defines a positive operator $T \in \mathcal{L}(l^p, l^1)$ as follows :

$$T(\{x_n\}_n) = \{a_n x_n\}_n.$$

Clearly the product $T \circ i_p$ is not a nuclear operator, q.e.d.

This answers to a question posed by Jameson [11].

The fact that an M-space cannot be isomorphic to a space $L^p(\mu)$ $(1 \leqslant p < \infty)$ was first remarked by Grothendieck [7].

4.8. *Remark.* The second dual of an aM-space is also an aM-space. In fact, let us consider an operator $T \in \mathcal{L}(E^{**}, L^1(\mu))$, $T \geqslant 0$. The restriction of T to E is a positive operator $S \in \mathcal{L}(E, L^1(\mu))$. Then S is absolutely summing and therefore ([8] ch. 1, Lemma 17) weakly compact, which implies that $T = S^{**}$. It remains to observe that the biadjoint of an absolutely summing operator is also absolutely summing ([8] ch. 1, Lemme 17, Corollaire; in the terminology of Grothendieck absolutely summing = semi-intégrale droite) q.e.d.

5. APPLICATION TO THE NUCLEAR VECTOR LATTICES

We first present a brief survey of the main properties of the p-absolutely summing operators. See Pietsch [26] for details.

Let E, F be two Banach spaces and $1 \leqslant p < \infty$.

5.1. Definition. An operator $T \in \mathcal{L}(E, F)$ is said to be p-absolutely summing if it verifies the following equivalent conditions :

$$(\pi_p.1) \qquad T(l^p[E]) \subset l^p\{F\}$$

$(\pi_p.2)$ There exists a positive Radon measure μ on the unit ball S^* of E^* such that :

$$\|Tx\| \leqslant \int |\langle x, x^* \rangle| \, d\mu \, (x^*)$$

for every $x \in E$. Instead of S^* we can consider here any weak*-closed subset $K \subset S^*$ such that $\|x\| = \sup_{x^* \in S^*} |\langle x, x^* \rangle|$ for every $x \in E$.

$(\pi_p.3)$ There exists a constant $C > 0$ such that for arbitrary $x_1, x_2, \ldots, x_n$ in E :

$$\sum_{i=1}^{n} \| Tx_i \|^p \leqslant C \sup_{x^* \in S^*} \left\{ \sum_{i=1}^{n} | \langle x_i, x^* \rangle |^p \right\}.$$

Denote by $\Pi_p(E, F)$ the vector space of all p-absolutely summing operators $T \in \mathcal{L}(E, F)$.

5.2. PROPOSITION [4]. *If E is a Banach space and $T \in \mathcal{L}(E, L^p(\mu))$ maps the unit ball of E into an order bounded subset of $L^p(\mu)$ then $T \in \Pi_p$ $(E, F)(1 \leqslant p < \infty)$.*

Proof. For $p=1$ this result was obtained by Grothendieck [8] ch. 1, Theoreme 11. The general case involves only slight modifications by using the lifting theorem in [10] instead of the Dunford-Pettis lifting theorem i.e. for every positive Radon measure μ defined on a locally compact Hausdorff space S there exists a function $\alpha : S \to (L^\infty(\mu))^*$ such that :

(i) $\| \alpha(s) \| = 1, \quad s \in S.$

(ii) The function $s \to \langle f, \alpha(s) \rangle$ is μ-measurable and μ-equal with f, whenever $f \in L^\infty(\mu)$.

Finally remark that for E an aM-space, $p = 1$ and $T \geqslant 0$ our assertion follows immediately from the conditions aM and (L^p, a) above.

5.3. *Remark.* Let E be an aM-space and $1 \leqslant p < \infty$. Then

$$T \in \mathcal{L}(E, L^p(\nu)), \quad T \geqslant 0 \text{ implies } T \in \Pi_p(E, L^p(\nu)),$$

and thus we can reformulate Theorem 3.1 above as follows :

A Banach lattice E is order isomorphic to an abstract L^p-space if, and only if, it verifies the following two conditions :

$(L^p.a')$ For every aM-space Z we have :

$$T \in \mathcal{L}(Z, E), \quad T \geqslant 0 \text{ implies } T \in \Pi_p(Z, E)$$

$(L^p.b)$ $$\| \Sigma x_i \|^p \leqslant \gamma \Sigma \| x_i \|^p,$$

for every finite family $\{x_i\}_i$ of disjoint elements of E.

We shall prove only that $(L^p.a') \Rightarrow (L^p.a)$. In fact, let us consider a sequence $\{x_n\}_n \in l^1[E]$, $x_n \geqslant 0$. It generated a positive operator $T \in \mathcal{L}(c_0, E)$ given by :

$$T(\{\alpha_n\}) = \sum_{n=1}^{\infty} \alpha_n x_n .$$

From hypothesis it follows that T is p-absolutely summing which implies that $\{x_n\}_n \in l^p\{E\}$. On the other hand it was remarked by Pietsch [26] Satz 17, that every p-absolutely summing operator is weakly compact, which implies (see the proof of 4.6 above) that $\{x_n\}_n \in l^1(E)$, q.e.d.

The next result was obtained by Pietsch [26] Theorem 4 (see also [8] ch. 1, Théorème 14 for $p = 1$) :

[4] See also Kwapien S., On a theorem of L. Schwartz and its applications to absolutely summing operators, Studia Math., 38 (1970), 193—201.

5.4. THEOREM. *Let* $1 \leqslant p,\ q < \infty$, E, F, G *Banach spaces,* $T \in \Pi_p(E, F)$ *and* $S \in \Pi_q(F, G)$. *Then :*

a) *If* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leqslant 1$, $S \circ T \in \Pi_r(E, G)$,

b) *If* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \geqslant 1$, $S \circ T \in \Pi_1(E, G)$.

In other words the product of $2p$ p-absolutely summing operators is a nuclear operator.

5.5. *Normed spaces which may be associated canonically to a locally convex space* E.

Let p be a continuous semi norm on E. Denote :

$$N_p = \{x \in E;\ p(x) = 0\}.$$

Then $E/N_p = E_p$ is a normed space with respect to p. Given two continuous semi norms p and q, $p \leqslant q$, there is defined a canonical mapping $E_q \to E_p$ which is continuous. We can extend this mapping to a mapping from $\hat{E}_q$ to $\hat{E}_p$. Here the cap denotes the completion.

For U an absolutely convex closed neighbourhood of 0 denote by P_U the semi norm associated and by E_U the normed space E_{p_U}.

For every absolutely convex bounded subset $A \subset E$ denote by E_A the vector space generated by A and endowed with the norm :

$$\|x\|_A = \inf\{\lambda > 0;\quad x \in \lambda A\}.$$

There exists a natural duality between the spaces E_U and E_A.

5.6. DEFINITION. A locally convex vector lattice E is said to be locally (respectively locally*) an L^1-space if there exists a fundamental system $\mathcal{U}$ of absolutely convex closed solid neighbourhoods of 0 (bounded subsets of E) such that the Banach lattices $\hat{E}_A$ $A \in \mathcal{A}$ be all L^1-spaces.

Similar notions are introduced for M-and L^p-spaces.

An important class of locally convex vector lattices which are locally M-spaces and L^p-spaces $(1 \leqslant p < \infty)$ is the class of all nuclear lattices. This follows easily by using the techniques in [25] especially 6.1.2 and 6.1.3.

Recall that a locally convex vector space E is said to be nuclear (nuclear*) if for every continuous semi-norm p (for every absolutely convex bounded subset $A \subset E$) there exists a continuous semi norm q, $p \leqslant q$ (an absolutely bounded subset $B \subset E$, $A \subset B$) such that the canonical mapping $\hat{E}_q \to \hat{E}_p$ (respectively $\hat{E}_A \to \hat{E}_B$) be absolutely summing. For a detailed account of the nuclear vector spaces see [8] and [25].

The following result holds :

5.7. THEOREM. *Let E be a locally convex vector lattice. Then E is a nuclear (nuclear*) vector lattice if, and only if, it satisfies the following conditions :*

(N_1) *E is locally (locally*) an M-space,*

(N_2) *E is locally (locally*) an L^p-space for some $1 \leqslant p < \infty$.*

Proof. It suffices to prove only the sufficiency. Or this follows from 5.3 and 5.4 above.

For $p = 1$ this result was earlier establishedd by Popa [27].

APPENDIX CONCERNING THE ORDER ON BANACH SPACES

The aim of this section is to prove the existence of a Banach space which does not admit any structure of Banach lattice. A result of Hogbe Nlend [9] asserts that every Banach space is the strong dual of a suitable nuclear space and thus the result above and the Komura - Koshi's characterization for nuclear lattices in [15] imply together that there exist nuclear spaces which are not isomorphic to a space of generalized sequences. However the basis problem in the Frechet nuclear spaces (i.e. Need every Fréchet nuclear space be isomorphic to a space of sequences?) remains still open.

That the order problem has a negative answer for the locally convex vector spaces was earlier remarked by Schaeffer who proved [28] that a reflexive lattice is necessarily topological complete. See [14] for an example of non-complete Montel space.

Our basic tool is the study of Banach lattices whose topological dual is l^1. We need a useful property of l^1 namely :

LEMMA 1.[5] *The Banach space l^1 admits a unique (up to an algebraic topologic lattice isomorphism) structure of order complete Banach lattice.*

Proof. In fact let us consider l^1 endowed with a structure of complete Banach lattice. Since l^1 is a separable dual it follows from 2.4 above that its topology is order continuous and therefore each order interval in l^1 must be relatively weakly compact (2.3 above). On the other hand every relatively weakly compact subset of l^1 is relatively compact. From [33] Theorem 1 it follows that the order considered on l^1 is discrete i.e. it is the coordinatewise order associated to a suitable unconditional basis of l^1. Or it was remarked in [18] Theorem 1 that l^1 has (up to equivalence) a unique unconditional normalized basis.

Recall that a basis $\{x_n\}_n$ is called normalized if $\|x_n\| = 1$ for every $n \in \mathbb{N}$. Two bases $\{x_n\}_n$ and $\{y_n\}_n$ of a Banach space X are said to be equivalent if there exists an invertible linear operator $T \in \mathcal{L}(X, Y)$ such that $Tx_n = y_n$ for every $n \in \mathbb{N}$.

From Lemma 1 above we deduce the following :

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[5] Added in proof, September 15, 1974. After this paper has been sent to the printer we found this property stated without proof in Classical Banach Spaces (Lectures notes in Math., Springer-Verlag n° 338, 1973) by Lindenstrauss J. and Tzafriri L.

LEMMA 2. *Let X be a Banach lattice whose topological dual is l^1. Then X is isometric to a subspace of a space $C(K)$ consisting of all the functions $f \in C(K)$ satisfying a set Ω of relations of the form :*

$$f(k_a^1) = \lambda_a f(k_a^2),$$

where $k_a^1, k_a^2 \in K$, $\lambda_a \in \mathbb{R}$, $a \in \Omega$. In other words X is a G-space in the terminology of [16].

Proof. From Lemma 1 it follons that there exists an algebraic topologic lattice isomorphism $\varphi : l^1 \to X^*$. Denote by $i_X : X \to X^{**}$ the canonical mapping. Then $X^{**} = l^\infty$ as Banach spaces and $Z = \varphi^* \circ i_X(X)$ is a Banach sub-lattice of l^∞. A classical result due to Kakutani [13] implies that Z consists precisely from all the continuous functions f defined on the Stone-Cech compactification of $\mathbb{N}$ and satisfying a set Ω of relations of the form

$$\varphi^* \circ i_X(x)(k_a^1) = \lambda_a \varphi^* \circ i_X(x)(k_a^2)$$

for every $x \in X$. On the other hand φ is an algebraical isomorphism and our result follows.

We can establish easily the existence of Banach spaces which do not admit any structure of Banach Lattice. In fact, it was remarked by Lindenstrauss [16] that there exist Banach spaces whose topological dual is l^1 and which are not G-space e.g. the space (with the sup norm) of all convergent real sequences $\{x_n\}_n$ such that $\lim_{n\to\infty} x_n = \frac{x_1 + x_2}{2}$.

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